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Uniqueness of Best Chebyshev Approximation on Subsets

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Let X be a compact space. For Y a compact subset of X, let $C(Y)$ be the space of real continuous functions on Y. For $g \in C(Y)$ define

$$\| g \|_Y = \sup\{| g(x)| : x \in Y\}, \qquad \| g \| = \| g \|_X .$$

Let $\{\phi_1 ,..., \phi_n\}$ be a linearly independent subset of $C(X)$ and define

$$L(A, x) = \sum_{k=1}^{n} a_k \phi_k(x).$$

The Chebyshev approximation problem on Y is: given $f \in C(Y)$ to find A^* minimizing $\|f - L(A, .)\|_Y$. Such a parameter A^* is called best. This approximation problem is studied in [1, Chapter 3; 2; 3, Chapter 12].

DEFINITION. Let $\{X_k\}$ be a sequence of compact subsets of X. We say $\{X_k\} \to X$ if for any $x \in X$, there is an $\{x_k\} \to x$, $x_k \in X_k$. A well known result [1, p. 87] is the following.

THEOREM. *Let f have a unique best approximation $L(A,.)$ to f on X. Let $\{X_k\} \to X$ and $L(A_k , .)$ be best to f on X_k, then* $\| L(A,.) - L(A_k , .)\| \to 0$.

The related problem we wish to consider is whether f having a unique best approximation on X implies that best approximations are unique on all sufficiently dense subsets. This is obviously the case if f is an approximant $L(A,.)$: we henceforth assume that f is not an approximant.

1. SUFFICIENT CONDITIONS FOR UNIQUENESS

Let us fix f and define

$$M(Y, A) = \{x : |f(x) - L(A, x)| = \|f - L(A,.)\|_Y , \quad x \in Y\}.$$

By continuity of $|f - L(A, .)|$ and compactness of Y, $M(Y, A)$ is a closed nonempty set. The set $M(Y, A)$ plays a critical role in the characterization of best approximations and in uniqueness. The following definitions are due to Lawson [2, pp. 22–23].

DEFINITION. A subset W of Y is an *error-determining set* (ED set) for f on Y if

$$\inf\{\sup\{|f(x) - L(A, x)| : x \in Y\} : A \in E_n\}$$
$$= \inf\{\sup\{|f(x) - L(A, x)| : x \in W\} : A \in E_n\}.$$

An irreducible error determining set (IED set) for f on Y is an ED set for f on Y which has no proper subset which is an ED set on Y.
IED sets are called "critical point sets" by Rice [3, p. 233]. It is a consequence of (i) the characterization result of Lawson [2], or (ii) the characterization result of Cheney [1, p. 73] and the theorem of Caratheodory [1, p. 17] that an IED set exists, contains at most $n + 1$ points and is a subset of $M(Y, A)$ for all A best on Y.

LEMMA. *Best approximations to f agree on any IED set for f.*

Proof. Let A, B be best to f and let W be a set on which $L(A, .)$ and $L(B, .)$ differ, say at the point x. By convexity of the set of best coefficients, $(A + B)/2$ is also best. Further $x \notin M(Y, (A + B)/2)$. Hence x cannot be in an IED set for f.

THEOREM. *A sufficient condition for best approximations to f to be unique is that the set of approximants be a Haar subspace of dimension n on an* IED *set.*

Proof. Let W be an IED set on which $\{L(A, .) : A \in E_n\}$ is a Haar subspace of dimension n. If W contained n or fewer points, an approximation could be selected taking any desired value on W. Hence W contains $n + 1$ points. As all best approximations agree on W, the Haar condition implies that a best approximation is unique.

COROLLARY. *A sufficient condition for best A to be unique on Y is that the family of approximations be a Haar subspace of dimension n on $M(Y, A)$.*

2. UNIQUENESS ON SUBSETS

If f has a unique best approximation, there may exist $\{X_k\} \to X$ such that f does not have a unique best approximation on X_k. This can occur even if the set of approximants is a Haar subspace of dimension n on an IED set for f.

EXAMPLE. Let $X = [-1, 1]$,

$$\phi(x) = 0 \qquad x \leqslant 0$$
$$= x \qquad x > 0,$$

$L(a, x) = a\phi(x)$, and

$$f(x) = \cos(2\Pi x) \qquad 0 \leqslant x \leqslant 1$$
$$= 1 + x \qquad -1 \leqslant x \leqslant 0.$$

We have $M(X, 0) = \{0, \frac{1}{2}, 1\}$ and since $\phi(0) = 0$, 0 is best. The set $\{\frac{1}{2}, 1\}$ is an IED set and the set of approximants is a Haar subspace of dimension 1 on $\{\frac{1}{2}, 1\}$. Hence 0 is a unique best approximation. We can select Y with density arbitrarily small such that $M(Y, 0) = \{0\}$ and 0 is a nonunique best approximation to f on Y.

However, if the set of approximants is a Haar subspace of dimension n on the error extrema of the best approximation, best approximations on all sufficiently dense subsets must be unique.

THEOREM. *Let A be best to f on X and the set of approximations be a Haar subspace of dimension n on $M(X, A)$. There exists $\epsilon > 0$ such that if the density of Y in X is less than ϵ, a best approximation to f on Y is unique.*

Proof. Suppose not, then there is a sequence $\{X_k\} \to X$ such that f does not have a unique best approximation on X_k. Let $\{A_k\}$ be best on X_k. A is unique by the preceeding corollary. It follows by the first theorem that $\{A_k\} \to A$. By nonuniqueness of A_k, there exists an IED set Y_k for f on X_k on which the Haar condition fails. By taking a subsequence if necessary we can assume that all IED sets Y_k contain the same number of points, say m points. The sequence Y_k of m-tuples of elements of a compact set has an accumulation point Y, assume $\{Y_k\} \to Y$. For given $x \in X$, define

$$I(x) = (\phi_1(x),..., \phi_n(x)).$$

As Y_k is an error-determining set for f on X_k, 0 is in the convex hull of $\{(f(x) - L(A_k, x)). I(x) : x \in Y_k\}$ by the characterization theorem [1, p. 73]. As $f - L(A_k, .) \to f - L(A, .)$ and $\{Y_k\} \to Y$, we have by continuity that 0 is in the convex hull of $\{(f(x) - L(A, x)) \cdot I(x): x \in Y\}$. Let $x \in Y$ and suppose $|f(x) - L(A, x)| < \|f - L(A, .)\|$. Then there exists $\epsilon > 0$ such that

$$|f(x) - L(A, x)| < \|f - L(A, .)\| - \epsilon$$

and hence a neighbourhood N of x such that

$$|f(y) - L(A, y)| < \|f - L(A, .)\| - \epsilon \qquad y \in N.$$

For all k sufficiently large, we have

$$|f(y) - L(A_k, y)| < \|f - L(A_k, .)\|_{X_k} - \epsilon/2 \qquad y \in N.$$

But this contradicts the existence of a sequence $\{y_k\} \to x$, $y_k \in M(X_k, A_k)$. Hence if $x \in Y$, $x \in M(X, A)$. It then follows from the characterization theorem that Y is an ED set for f on X and therefore contains an IED set Y' for f on X. As $Y' \subset M(X, A)$, the Haar condition is satisfied on Y'. If Y' has n or fewer points, approximants could be selected to agree with f on Y'. This would contradict Y' being an ED set so Y' has $n + 1$ points. Hence Y has $n + 1$ distinct points, call them $x_0, \ldots, x_n$. There exist sequences $\{x_i^k\}$, $i = 0, \ldots, n$ such that $x_i^k \in Y_k$ and $\{x_i^k\} \to x_i$, $i = 0, \ldots, n$. Let the Haar condition fail on the n point set $\{Y_k\} \sim x_{j(k)}^k$. There is an integer j in $0, \ldots, n$ such that $j(k)$ is j infinitely often. By taking a subset of $\{Y_k\}$ if necessary we can assume that $j(k) = j$ for all k. The Vandermonde determinant of the basis functions evaluated at the points $\{Y_k\} \sim x_j^k$ is then zero for all k, hence by continuity the Vandermonde determinant of the basis functions evaluated at the points $\{Y\} \sim x_j$ is zero. The Haar condition then fails on the n point subset $\{Y\} \sim x_j$ of $M(X, A)$, contradicting the hypothesis of the theorem.

The theorem is proven. The subset result becomes

THEOREM. *Let the family of approximations be a Haar subspace of dimension n on $M(X, A)$. Let $\{X_k\} \to X$. Then for all k sufficiently large f has a unique best approximation $L(A_k, .)$ on X_k and $\{L(A_k, .)\}$ converges uniformly to $L(A, .)$, the unique best approximation to f on X.*

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